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Moment inequalities and convergence rates in the strong laws for ρ^-- mixing random fields

Guang-hui Cai
Department of Mathematics and Statistics, Zhejiang Gongshang University, Hangzhou 310035, P. R. China
E-mail: cghzju@163.com

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By using a Rosenthal type inequality established in this paper and inspired by Berberan-Santos et al., J. Math. Chem. 37 (2005) 101–115, the complete convergence rates in the strong laws for ρ^- - mixing random fields are discussed. The result obtained extends the result of Ko et al., Commun. Korean Math. Soc. 19 (2004) 765–773.

KEY WORDS: convergence rates, rosenthal type inequality, ρ^--mixing, ρ^*-mixing, negatively associated, random fields

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1. Introduction

In 2001, Zhang gave the concept of ρ^--mixing random fields. As for ρ^--mixing random fields, Zhang [1] obtained a weak convergence. The concept of ρ^--mixing random fields see the following definition.

Definition 1 [1]. A field $\{X_i, i \in N^d\}$ s called ρ^--mixing if

$$\rho^-(s) = \sup\{(\rho^-(S,T); S, T \subset N^d, \operatorname{dist}(S,T) \geqslant s\} \to 0(s \to \infty),$$

where

$$\rho^-(S,T) = 0 \vee \sup\left\{\frac{\operatorname{Cov}\{f(X_i; i \in S), g(X_j; j \in T)\}}{[\operatorname{Var}\{f(X_i; i \in S)\}\operatorname{Var}\{g(X_j; j \in T)\}]^{1/2}}; f, g \in \mathcal{C}\right\}.$$

In 2005, Berberan-Santos, et al. [2] have discussed the classical and the quantum mechanical description of a one-dimensional motion of a particle in the presence of a gravitational field. Their attention is centered on the evolution of classical and quantum mechanical position probability distribution function. The classical case has been compared with three different quantum cases: (a) a quantum stationary case, (b) a quantum non-stationary zero approximation case, where the wave packet has the shape of the first eigenfunction, and (c) a quantum non-stationary general case, where the wave packet is a superposition of stationary states.

In 2004, Ko et al. [3] obtained strong laws of large numbers for asymptotically quadrant independent random fields. In this paper, inspired by Berberan-Santos et al. [2] and by using a Rosenthal type inequality established in this paper, the complete convergence rates in the strong laws for ρ^--mixing random fields are discussed. The result obtained extends the result of Ko et al. [3].

2. Theorems and proof

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n \ll b_n$ will mean $a_n \leqslant Cb_n$.

Lemma 1. [1]. Assume $\{X_i, i \in N^d\}$ be a ρ^--mixing random field, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geqslant 2$ and every $i \in N^d$. Then there exists $C = C(p, \rho^-(1))$, such that

$$E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|^p \leqslant C\left\{\left(E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|\right)^p + \sum_{i=1}^{n} E\,|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}. \tag{1}$$

Lemma 2. Assume $\{X_i, i \in N^d\}$ be a ρ^--mixing random field, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geqslant 2$ and every $i \in N^d$. Then there exists $C = C(p, \rho^-(n))$, such that

$$E\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|^p \leqslant C\left\{\sum_{i=1}^{n} E|X_i|^p + (\log_2 n)^p\left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}. \tag{2}$$

Proof. The proof is similar to the proof of corollary 2 in Peligrad and Gut [4]. □

Lemma 3. Assume events $A_1, A_2, \ldots, A_n$ satisfy $\mathrm{Var}(\sum_{k=1}^{n} I_{A_k}) \leqslant K\sum_{k=1}^{n} P(A_k)$, then

$$\left(1 - P\left(\bigcup_{j=1}^{n} A_j\right)\right)^2 \sum_{k=1}^{n} P(A_k) \leqslant KP\left(\bigcup_{j=1}^{n} A_j\right).$$

Proof. Let $\beta = 1 - P\left(\bigcup_{k=1}^{n} A_k\right)$.
If $\beta = 0$, then $\left(1 - P\left(\bigcup_{j=1}^{n} A_j\right)\right)^2 \sum_{k=1}^{n} P(A_k) \leqslant K P\left(\bigcup_{j=1}^{n} A_j\right)$ is obvious.
If $\beta > 0$, by Var $\left(\sum_{k=1}^{n} I_{A_k}\right) \leqslant K \sum_{k=1}^{n} P(A_k)$, then

$$\begin{aligned}\sum_{j=1}^{n} P(A_j) &= \sum_{j=1}^{n} P\left\{A_j \bigcap \left(\bigcup_{i=1}^{n} A_i\right)\right\}\\ &= E\left\{\sum_{j=1}^{n}(I_{A_j} - P(A_j)) I_{\{\bigcup_{i=1}^{n} A_i\}}\right\} + \sum_{j=1}^{n} P(A_j)(1-\beta)\\ &\leqslant \left(\operatorname{Var}\left(\sum_{j=1}^{n} I_{A_j}\right) P\left(\bigcup_{k=1}^{n} A_k\right)\right)^{1/2} + \sum_{j=1}^{n} P(A_j)(1-\beta)\\ &\leqslant \left\{K \sum_{j=1}^{n} P(A_j) P\left(\bigcup_{k=1}^{n} A_k\right)\right\}^{1/2} + \sum_{j=1}^{n} P(A_j)(1-\beta)\\ &\leqslant \frac{K}{2\beta} P\left(\bigcup_{k=1}^{n} A_k\right) + \left(\frac{\beta}{2} + 1 - \beta\right)\sum_{j=1}^{n} P(A_j).\end{aligned}$$

Thus

$$\sum_{j=1}^{n} P(A_j) \leqslant \frac{K}{2\beta} P\left(\bigcup_{k=1}^{n} A_k\right) + \left(1 - \frac{\beta}{2}\right)\sum_{j=1}^{n} P(A_j).$$

So we have

$$\sum_{j=1}^{n} P(A_j) \leqslant \frac{K}{\beta^2} P\left(\bigcup_{k=1}^{n} A_k\right).$$

Now we complete the proof of Lemma 3. □

Lemma 4 [1]. Assume $\{X_i, i \in N^d\}$ be a ρ^--mixing random field, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geqslant 2$ and every $i \in N^d$. Then there exists $C = C(p)$, such that

$$E\left|\sum_{n\in S} X_n\right|^p \leqslant C\left\{\sum_{n\in S} E|X_n|^p + \left(\sum_{n\in S} E|X_n|^2\right)^{p/2}\right\}, \quad \forall S \subset N^d.$$

Theorem 1. Let $\{X, X_i, i \in N^d\}$ be identically distributed ρ^--mixing random field, $\alpha > \frac{1}{2}$, $p\alpha > 1$, and suppose that $EX = 0$ for $\alpha \leqslant 1$. $S_n = \sum_{1\leqslant i\leqslant n} X_i$, and

$$E|X|^p \log^{d-1}(|X|) < \infty. \tag{3}$$

Then

$$\forall \varepsilon > 0, \sum_n |n|^{p\alpha-2} P\left(\max_{1\leqslant j\leqslant n} |S_j| > \varepsilon |n|^\alpha\right) < \infty. \tag{4}$$

Proof. $\forall i \in N^d$, define $X_i^{(n)} = X_i I(|X_i| \leqslant |n|^\alpha) + |n|^\alpha I(X_i > |n|^\alpha) - |n|^\alpha I(X_i < -|n|^\alpha)$,

$S_j^{(n)} = \sum_{1\leqslant i\leqslant j}(X_i^{(n)} - EX_i^{(n)})$. Then $\forall \varepsilon > 0$,

$$\begin{aligned} P\left(\max_{1\leqslant j\leqslant n} |S_j| > \varepsilon |n|^\alpha\right) &\leqslant P\left(\max_{1\leqslant j\leqslant n} |X_j| > |n|^\alpha\right) \\ &+ P\left(\max_{1\leqslant j\leqslant n} |S_j^{(n)}| > \varepsilon |n|^\alpha - \max_{1\leqslant j\leqslant n}\left|\sum_{1\leqslant i\leqslant j} EX_i^{(n)}\right|\right). \end{aligned} \tag{5}$$

First we show that

$$|n|^{-\alpha} \max_{1\leqslant j\leqslant n}\left|\sum_{1\leqslant i\leqslant j} EX_i^{(n)}\right| \to 0, \text{ as } |n| \to \infty. \tag{6}$$

In fact, (i) If $\alpha > 1$, $p \geqslant 1$, then

$$\begin{aligned} &|n|^{-\alpha} \max_{1\leqslant j\leqslant n}\left|\sum_{1\leqslant i\leqslant j} EX_i^{(n)}\right| \\ &\leqslant |n|^{-\alpha} \sum_{1\leqslant i\leqslant n} E|X_i| I(|X_i| \leqslant |n|^\alpha) + \sum_{1\leqslant i\leqslant n} P(|X_i| > |n|^\alpha) \\ &\leqslant |n|^{1-\alpha} E|X| I(|X| \leqslant |n|^\alpha) + |n| P(|X| > |n|^\alpha) \\ &\leqslant |n|^{1-\alpha} E|X| I(|X| \leqslant |n|^\alpha) + |n| \frac{E|X|^p}{|n|^{\alpha p}} \\ &\ll |n|^{1-\alpha} E|X| I(|X| \leqslant |n|^\alpha) + |n|^{1-\alpha p} \\ &\to 0. \end{aligned} \tag{7}$$

(ii) If $\alpha > 1$, $p < 1$, using (7), then

$$|n|^{-\alpha} \max_{1\leqslant j\leqslant n}\left|\sum_{1\leqslant i\leqslant j} EX_i^{(n)}\right| \leqslant |n|^{1-\alpha} \sum_{k=1}^{|n|} E|X| I(k-1 < |X|^{\frac{1}{\alpha}} \leqslant k) + |n|^{1-\alpha p}.$$

Using $p < 1, \alpha p > 1,$ then

$$
\begin{aligned}
&\sum_{k=1}^{\infty} k^{1-\alpha} E|X| I(k-1 < |X|^{\frac{1}{\alpha}} \leqslant k) \\
&\quad \leqslant \sum_{k=1}^{\infty} k^{1-\alpha} E|X|^p I(k-1 < |X|^{\frac{1}{\alpha}} \leqslant k) k^{\alpha(1-p)} \\
&\quad = \sum_{k=1}^{\infty} k^{1-\alpha p} E|X|^p I(k-1 < |X|^{\frac{1}{\alpha}} \leqslant k) \\
&\quad \leqslant \sum_{k=1}^{\infty} E|X|^p I(k-1 < |X|^{\frac{1}{\alpha}} \leqslant k) \\
&\quad = E|X|^p < \infty.
\end{aligned}
$$

By Kronecker lemma, we get $|n|^{1-\alpha} \sum_{k=1}^{|n|} E|X| I(k-1 < |X|^{\frac{1}{\alpha}} \leqslant k) \to 0, \ |n| \to \infty$, So

$$
|n|^{-\alpha} \max_{1 \leqslant j \leqslant n} \left| \sum_{1 \leqslant i \leqslant j} E X_i^{(n)} \right| \to 0, \quad |n| \to \infty. \tag{8}
$$

(iii) If $\frac{1}{2} < \alpha \leqslant 1$, noting that $p \geqslant 1$, $p\alpha > 1$ and $EX = 0$ it follows that

$$
\begin{aligned}
&|n|^{-\alpha} \max_{1 \leqslant j \leqslant n} \left| \sum_{1 \leqslant i \leqslant j} E X_i^{(n)} \right| \\
&\quad \leqslant |n|^{-\alpha} \sum_{1 \leqslant i \leqslant n} E|X_i| I(|X_i| > |n|^{\alpha}) + \sum_{1 \leqslant i \leqslant n} P(|X_i| > |n|^{\alpha}) \\
&\quad \leqslant |n|^{1-\alpha p} E|X|^p I(|X| > |n|^{\alpha}) + |n| P(|X| > |n|^{\alpha}) \\
&\quad \leqslant |n|^{1-\alpha p} E|X|^p I(|X| > |n|^{\alpha}) + |n| \frac{E|X|^p}{|n|^{\alpha p}} \\
&\quad \ll |n|^{1-\alpha p} E|X|^p I(|X| > |n|^{\alpha}) + |n|^{1-\alpha p} \\
&\quad \to 0.
\end{aligned} \tag{9}
$$

From equations (7)–(9), which imply (6).
From equations (5) and (6) it follows that for $|n|$ large enough

$$
P\left(\max_{1 \leqslant j \leqslant n} |S_j| > \varepsilon |n|^{\alpha} \right) \leqslant \sum_{1 \leqslant j \leqslant n} P(|X_j| > |n|^{\alpha}) + P\left(\max_{1 \leqslant j \leqslant n} |S_j^{(n)}| > \frac{\varepsilon}{2} |n|^{\alpha} \right).
$$

Hence we need only to prove that

$$\mathrm{I} =: \sum_{n} |n|^{\alpha p-2} \sum_{1\leqslant j\leqslant n} P(|X_j| > |n|^{\alpha}) < \infty, \tag{10}$$

$$\mathrm{II} =: \sum_{n} |n|^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n} |S_j^{(n)}| > \frac{\varepsilon}{2}|n|^{\alpha}\right) < \infty. \tag{11}$$

From the fact that $E|X|^p \log^{d-1}(|X|) < \infty$, it follows easily that

$$\begin{aligned} \mathrm{I} &\ll \sum_{m=1}^{\infty} m^{\alpha p-1} \log^{d-1}(m) P(|X| > m^{\alpha}) \\ &\ll E|X|^p \log^{d-1}(|X|) < \infty. \end{aligned} \tag{12}$$

By Lemma 2, it follows that

$$\begin{aligned} \mathrm{II} &\ll \sum_{n} |n|^{\alpha p-2-\alpha q} E \max_{1\leqslant j\leqslant n} |S_j^{(n)}|^q \\ &\ll \sum_{n} |n|^{\alpha(p-q)-2} \left\{ \sum_{1\leqslant j\leqslant n} E|X_j^{(n)}|^q + (\log_2 |n|)^q \left(\sum_{1\leqslant j\leqslant n} E|X_j^{(n)}|^2 \right)^{q/2} \right\} \\ &=: \mathrm{II}_1 + \mathrm{II}_2. \end{aligned} \tag{13}$$

where $q = 2$ if $0 < p < 2$; $q = \frac{\alpha p-1}{\alpha-\frac{1}{2}} + p$, if $p \geqslant 2$. One can show that

$$\begin{aligned} \mathrm{II}_1 &= \sum_{n} |n|^{\alpha(p-q)-1} E|X|^q I(|X| \leqslant |n|^{\alpha}) \\ &\ll \sum_{m=1}^{\infty} m^{\alpha(p-q)-1} \log^{d-1}(m) E|X|^q I(|X| \leqslant m^{\alpha}) \\ &= \sum_{m=1}^{\infty} m^{\alpha(p-q)-1} \log^{d-1}(m) \sum_{k=1}^{m} E|X|^q I(k-1 < |X|^{\frac{1}{\alpha}} \leqslant k) \\ &= \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} m^{\alpha(p-q)-1} \log^{d-1}(m) E|X|^q I(k-1 < |X|^{\frac{1}{\alpha}} \leqslant k) \\ &\ll \sum_{k=1}^{\infty} k^{\alpha p} \log^{d-1}(k) P(k-1 < |X|^{\frac{1}{\alpha}} \leqslant k) \\ &\ll E|X|^p \log^{d-1}(|X|) < \infty. \end{aligned} \tag{14}$$

If $0 < p < 2$, then let $q = 2$, it follows that $\mathrm{II}_2 = \mathrm{II}_1 < \infty$.

When $p \geqslant 2$, by $q = \frac{\alpha p-1}{\alpha-\frac{1}{2}} + p$, we have

$$\begin{aligned} \mathrm{II}_2 &\ll \sum_n |n|^{\alpha(p-q)-2+\frac{q}{2}} (\log_2 |n|)^q \\ &\ll \sum_{m=1}^{\infty} m^{\alpha(p-q)-2+\frac{q}{2}} (\log_2 m)^q \\ &= \sum_{m=1}^{\infty} m^{\alpha p-2-q(\alpha-\frac{1}{2})} (\log_2 m)^q \\ &= \sum_{m=1}^{\infty} m^{-1-p(\alpha-\frac{1}{2})} (\log_2 m)^q < \infty. \end{aligned} \tag{15}$$

Putting equations (14) and (15) into (13) yields II $< \infty$. Now we complete the proof of theorem 1. □

Corollary 1. Let $p = 2, \alpha = 1$, by Theorem 1 and Borel–Cantelli Lemma. Then

$$\frac{S_n}{n} \to 0 \ a.s.$$

Remark 1. Corollary 1 generalizes the Theorem 2.1 of Ko [3] to ρ^--mixing random fields.

Theorem 2. Let $\{X, X_i, i \in N^d\}$ be identically distributed ρ^--mixing random field, $\alpha > \frac{1}{2}$, $p\alpha \geqslant 1$, and suppose that $EX = 0$ for $\alpha \leqslant 1$. $S_n = \sum_{1 \leqslant i \leqslant n} X_i$, and

$$\forall \varepsilon > 0, \sum_n |n|^{p\alpha-2} P\left(\max_{1 \leqslant j \leqslant n} |S_j| > \varepsilon |n|^{\alpha}\right) < \infty. \tag{16}$$

Then

$$E|X|^p \log^{d-1}(|X|) < \infty. \tag{17}$$

Proof. By $\max_{1 \leqslant j \leqslant n} |X_j| \leqslant 2 \max_{1 \leqslant j \leqslant n} |S_j|$ and (16), we have

$$\sum_n |n|^{p\alpha-2} P\left(\max_{1 \leqslant j \leqslant n} |X_j| > 2\varepsilon |n|^{\alpha}\right) < \infty. \tag{18}$$

By $p\alpha \geqslant 1$ and (18), when $|n| \to \infty$, then

$$P\left(\max_{1 \leqslant j \leqslant n} |X_j| > 2\varepsilon |n|^{\alpha}\right) \to 0. \tag{19}$$

By $\{X_i, i \in N^d\}$ is a NA random field, then $\{I(X_j > x) - E[I(X_j > x)], j \in N^d\}$ and $\{I(X_j < -x) - E[I(X_j < -x)], j \in N^d\}$ are all NA random fields.
By lemma 4, then

$$\begin{aligned}
&\mathrm{Var}\left[\sum_{j=1}^{n} I\left(|X_j| > 2\varepsilon|n|^{\alpha}\right)\right] \\
&\leqslant 2\left\{\mathrm{Var}\left[\sum_{j=1}^{n} I(X_j > 2\varepsilon|n|^{\alpha})\right] + \mathrm{Var}\left[\sum_{j=1}^{n} I(X_j < -2\varepsilon|n|^{\alpha})\right]\right\} \\
&\leqslant 4C\left[\sum_{j=1}^{n} P(X_j > 2\varepsilon|n|^{\alpha}) + \sum_{j=1}^{n} P(X_j < -2\varepsilon|n|^{\alpha})\right] \\
&= 4C\sum_{j=1}^{n} P(|X_j| > 2\varepsilon|n|^{\alpha}). \qquad (20)
\end{aligned}$$

By equations (19) and (20) and lemma 3, when $|n| \to \infty$, then

$$\sum_{1\leqslant j\leqslant n} P\left(|X_j| > 2\varepsilon|n|^{\alpha}\right) \ll P\left(\max_{1\leqslant j\leqslant n} |X_j| > 2\varepsilon|n|^{\alpha}\right). \qquad (21)$$

By equations (18) and (21), then $\sum_n |n|^{p\alpha-1} P(|X| > 2\varepsilon|n|^{\alpha}) < \infty$.
Thus $\sum_{m=1}^{\infty} m^{p\alpha-1} \log^{d-1}(m) P(|X| > 2\varepsilon m^{\alpha}) < \infty$.
Then $E|X|^p h(|X|^{\frac{1}{\alpha}}) \log^{d-1}(|X|) < \infty$. Now we complete the prove of Theorem 2. □

Because ρ^--mixing random fields are more general than NA field or ρ^*-mixing field. So we have the following two corollaries.

Corollary 2. Let $\{X, X_i, i \in N^d\}$ be identically distributed NA field, $\alpha > \frac{1}{2}$, $p\alpha > 1$, and suppose that $EX = 0$ for $\alpha \leqslant 1$. $S_n = \sum_{1\leqslant i\leqslant n} X_i$, and

$$E|X|^p \log^{d-1}(|X|) < \infty. \qquad (22)$$

Then

$$\forall\varepsilon > 0, \sum_n |n|^{p\alpha-2} P\left(\max_{1\leqslant j\leqslant n} |S_j| > \varepsilon|n|^{\alpha}\right) < \infty. \qquad (23)$$

Corollary 3. Let $\{X, X_i, i \in N^d\}$ be identically distributed ρ^*-mixing field, $\alpha > \frac{1}{2}$, $p\alpha > 1$, and suppose that $EX = 0$ for $\alpha \leqslant 1$. $S_n = \sum_{1\leqslant i\leqslant n} X_i$, and

$$E|X|^p \log^{d-1}(|X|) < \infty. \qquad (24)$$

Then

$$\forall \varepsilon > 0, \sum_{n} |n|^{p\alpha-2} P\left(\max_{1\leqslant j\leqslant n} |S_j| > \varepsilon|n|^{\alpha}\right) < \infty, \tag{25}$$

where the definitions of NA fields and ρ^*-mixing fields see the following.

Definition 2 [1]. A field $\{X_i, i \in N^d\}$ is called negatively associated (NA) if for every pair of disjoint subsets T_1,T_2 of N^d,

$$\mathrm{Cov}(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leqslant 0,$$

whenever f_1 and f_2 are coordinatewise increasing.

Definition 3 [5]. A field $\{X_i, i \in N^d\}$ is called ρ^*-mixing if

$$\rho^*(s) = \sup\{(\rho(S, T); S, T \subset N, \mathrm{dist}(S, T) \geqslant s\} \to 0 (s \to \infty),$$

where

$$\rho(S, T) = \sup\{ |E(f - Ef)(g - Eg)|/\|f - Ef\|_2\|g - Eg\|_2, f \in L_2(\sigma(S)), g \in L_2(\sigma(T))\}.$$

Corollary 4. Let $p = 2, \alpha = 1$, by corollary 2 and Borel–Cantelli lemma. Then

$$\frac{S_n}{n} \to 0 \ a.s.$$

Corollary 5. Let $p = 2, \alpha = 1$, by corollary 3 and Borel–Cantelli lemma. Then

$$\frac{S_n}{n} \to 0 \ a.s.$$

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